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Quantum speed limit for physical processes

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Abstract

The evaluation of the minimal evolution time between two distinguishable states of a system is important for assessing the maximal speed of quantum computers and communication channels. Lower bounds for this miminal time have been proposed for unitary dynamics. Here we derive an attainable lower bound valid for general physical processes, which is connected to the quantum Fisher information for time estimation. This result is used to delimit the minimal evolution time for typical noisy channels.

Introduction

Quantum mechanics imposes fundamental limits to the processing speed of any device as well as to the communication speed through any channel. Derivation of these basic limits usually assumes that such devices are noiseless, undergoing unitary evolutions [1–37]. The relevant (and often unwanted) influence of the environment on processing or information-transfering systems is thus frequently ignored. On the other hand, this influence, and in particular the decoherence speed, plays an essential role in fundamental physics, especially in the understanding of the quantum-to-classical transition [38]. Here we unify the description of both computation/communication speed and decoherence speed in a single framework, which deals with the maximal speed of evolution of quantum systems.

Although much work has been done on the subject since the first major result by Mandelstam and Tamm [1], scarce contributions [39–41] undertake nonunitary evolutions. In this Letter, we geometrically derive a general lower bound to evolution times, which depends on the quantum Fisher information $\mathcal{F}_Q(t)$ [42] for the problem of time estimation. Besides recovering, in the proper limits, previous findings such as the Mandelstam-Tamm bound [1], our result allows the study of experimentally more realistic open systems and the development of a systematic approach for tackling such nonunitary evolutions. We exemplify the usefulness of this bound by considering typical nonunitary quantum channels

General bound

Let $D(F_B)$ be any metric on the space of quantum states, assumed to depend only on the Bures fidelity F_B,

$$F_B(\hat{\rho}, \hat{\sigma}) := \left[\mathrm{tr}\left(\sqrt{\sqrt{\hat{\rho}}\hat{\sigma}\sqrt{\hat{\rho}}}\right)\right]^2. \quad (1)$$

We denote by $D(t_1, t_2)$ a distance, along this metric, between the states of the same system at times t_1 and t_2, i.e., a shorthand notation for $D\{F_B\left[\hat{\rho}(t_1), \hat{\rho}(t_2)\right]\}$. A bound on $D(0, \tau)$ is obtained by applying to this distance the triangle inequality, considering a division of the interval $(0, \tau)$ into infinitesimal pieces, and using the relation between the Bures fidelity and the quantum Fisher information [42],

$$F_B(t_1, t_2) = 1 - (t_2 - t_1)^2 \frac{\mathcal{F}_Q(t_1)}{4} + \mathcal{O}(t_2 - t_1)^3. \quad (2)$$

One thus gets a general bound on the distance, which depends on the quantum Fisher information $\mathcal{F}_Q(t)$, and is valid for general physical processes (see

Supplemental Material for detailed derivation),

$$\sqrt{\frac{2d^2D(F_B)/dF_B^2}{\left[dD(F_B)/dF_B\right]^3}}\Bigg|_{F_B\to 1} D(0,\tau) \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)}dt. \tag{3}$$

Notice that this bound is invariant by a rescaling $D' = kD$, and yields an implicit lower bound on the evolution time τ.

Although (3) is valid for any metric $D(F_B)$, we show in the Supplemental Material that the metric $D(F_B) = 2\arccos\sqrt{F_B}$ (arccos being defined on $[0,\pi]$ throughout the article) leads to the greatest lower bound on τ,

$$2\arccos\sqrt{F_B(0,\tau)} \leq \int_0^\tau \sqrt{\mathcal{F}_Q(t)}dt, \tag{4}$$

where the shorthand notation $F_B(t_1,t_2)$ for $F_B\left[\hat\rho(t_1),\hat\rho(t_2)\right]$ is used. One should note that the l.h.s. is the shortest distance between the two states $\hat\rho(0)$ and $\hat\rho(\tau)$, while the r.h.s. is the length of the actual path followed by the state of the system, under a given dynamics. This implies, as shown in Supplemental Material, that this bound is attained if and only if the evolution occurs on a geodesic.

For a unitary evolution, dictated by the operator $\hat U(t)$, the quantum Fisher information is given by $\mathcal{F}_Q(t) = 4\langle\Delta\hat H^2(t)\rangle/\hbar^2$ [43], where $\langle\Delta\hat H^2(t)\rangle$ is the variance in the initial state of an operator $\hat H(t)$ defined as

$$\hat H(t) := \frac{\hbar}{i}\frac{d\hat U^\dagger(t)}{dt}\hat U(t). \tag{5}$$

For a time-independent Hamiltonian $\hat H$, $\hat U(t) = e^{-iHt/\hbar}$, and $\hat H(t) = \hat H$. Equation (4) yields then the Mandelstam-Tamm bound.

For nonunitary evolutions, the quantum Fisher information $\mathcal{F}_Q(t)$ can be calculated by means of a purification procedure [44]. For each system of interest S, represented by the density operator $\hat\rho_S$, we assign an environment E. We consider the dynamics of $\hat\rho_S$ as resulting from a unitary evolution, corresponding to the operator $\hat U_{S,E}(t)$, of a pure state of the enlarged system $S+E$. The quantum Fisher information of $S+E$, which is an upper bound $\mathcal{C}_Q(t)$ to the quantum Fisher information of subsystem S, is given by four times the variance of $\hat H_{S,E}(t)/\hbar$, defined by (5). There are, in fact, infinitely many different evolutions of $S+E$ corresponding to the same evolution of subsystem S, each of those leading to a value of $\mathcal{C}_Q(t)$. This freedom is integrally expressed by writing the purified unitary evolution as $\hat u_E(t)\hat U_{S,E}(t)$, where $\hat u_E(t)$ is any unitary operator acting on E alone. The actual value of $\mathcal{F}_Q(t)$ for each time t coincides with the minimal upper bound $\mathcal{C}_Q(t)$ over all possible evolutions of the enlarged system, that is, with the minimization of $\mathcal{C}_Q(t)$ with respect to $\hat u_E(t)$. Since $\mathcal{C}_Q(t)$ is written in terms of a variance that only depends on $\hat u_E(t)$ through $\hat h_E(t)$,

$$\hat h_E(t) := \frac{\hbar}{i}\frac{d\hat u_E^\dagger(t)}{dt}\hat u_E(t), \tag{6}$$

the minimization is performed with respect to $\hat h_E(t)$. It should be noted that, since we are interested in nonunitary evolutions, the value $F_B = 0$ is typically not attained, so that the minimum time corresponds in this case to a low, albeit finite, value of F_B.

We consider now examples that illustrate the power and usefulness of our general result.

Amplitude-damping channel

Let S be a two-state system (states $\{|0\rangle,|1\rangle\}$), and E its environment, which is chosen to start on state $|0\rangle_E$. The amplitude-damping channel is described by the map

$$|0\rangle|0\rangle_E \to |0\rangle|0\rangle_E \tag{7a}$$

$$|1\rangle|0\rangle_E \to \sqrt{P(t)}|1\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E, \tag{7b}$$

where the state $|1\rangle_E$ is orthogonal to $|0\rangle_E$, and the time dependence of $P(t)$ reflects the damping dynamics. We consider here the paradigmatic exponential decay, with rate γ, so that $P(t) = e^{-\gamma t}$. We note that, for the above map, the environment can also be considered as a qubit. This channel, which corresponds to a *nonunitary* evolution of S, can be described by the unitary evolution operator of $S+E$,

$$\hat U_{S,E}(t) = \exp\left\{-i\Theta(t)(\hat\sigma_+\hat\sigma_-^{(E)} + \hat\sigma_-\hat\sigma_+^{(E)})\right\}, \tag{8}$$

where $\hat{\sigma}_\pm$ and $\hat{\sigma}_\pm^{(E)}$ are raising and lowering operators acting respectively on the system and environment qubits, and $\Theta(t) = \arccos\sqrt{P(t)}$.

The bound can be calculated by taking the variance $\langle\Delta\hat{H}^2(t)\rangle$ of $\hat{H}(t)$ obtained from (8) via (5) into (4). If S is initially in the ground state, the bound yields the trivial result $F_B = 1$, whereas for S initially in the excited state,

$$F_B \geq e^{-\gamma\tau}, \tag{9}$$

yielding a lower bound for the minimum time $\tau \geq (1/\gamma)\ln(1/F_B)$. The right-hand side of (9) is the actual fidelity corresponding to the channel in this case. This shows that this bound is saturated for every τ and therefore we have already chosen the best purification. Furthermore, the fact that the bound is saturated implies that the nonunitary evolution connecting the inital and final states is through a geodesic path in state space (which includes mixed states). This reasoning does not depend on the specifics of $P(t)$ and it can be shown (Supplemental Material) that the bound is always saturated for initially excited states and monotonically decreasing $P(t)$, which is the relevant behavior for decaying states [45]. Interestingly, a single-mode Jaynes-Cummings (nonmonotonic) dynamics also displays such behavior. In this case, S is a two-level atom and E is the field, restricted to at most one excitation. The evolution can be described by $\Theta(t) = gt$, g being the coupling constant, and obeys (7), with convenient choices of the branches of the square roots. Saturation is again observed for every τ, without need for optimization. These results are atypical, as optimization is usually necessary for obtaining the highest lower bound for τ.

Markovian dephasing

Let us now consider Markovian dephasing. System S is again a single qubit whose nonunitary evolution is described by a map that makes use, as before, of an ancilla qubit starting in state $|0\rangle_E$,

$$\begin{aligned} |0\rangle|0\rangle_E &\to e^{-i\omega_0 t}\left(\sqrt{P(t)}|0\rangle|0\rangle_E + \sqrt{1-P(t)}|0\rangle|1\rangle_E\right), \\ |1\rangle|0\rangle_E &\to e^{i\omega_0 t}\left(\sqrt{P(t)}|1\rangle|0\rangle_E - \sqrt{1-P(t)}|1\rangle|1\rangle_E\right), \end{aligned} \tag{10}$$

where $\hbar\omega_0$ is the energy difference between the qubit levels, $P(t) := (1+e^{-\gamma t})/2$, and γ is the phase-decay constant. This description, equivalent to the master equation approach, points to the following evolution operator:

$$\hat{U}_{S,E}(t) = e^{-i\omega_0 t\hat{Z}} e^{-i\arccos\sqrt{P(\gamma t)}\hat{Z}\hat{Y}^{(E)}}, \tag{11}$$

where $\hat{Z}$ and $\hat{Y}^{(E)}$ are Pauli operators acting on the system and on the environment qubits, respectively.

Let $\hat{\mathcal{H}}_{S,E}(t)$ be defined by applying (5) to the evolution operator $\hat{u}_E(t)\hat{U}_{S,E}(t)$; we now minimize $\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle$ with respect to $\hat{h}_E(t)$ from (6), taken here as the most general form for a 2x2 Hermitian operator:

$$\hat{h}_E(t) = \alpha(t)\hat{X}^{(E)} + \beta(t)\hat{Y}^{(E)} + \delta(t)\hat{Z}^{(E)}, \tag{12}$$

where the identity is omitted for not changing $\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle$. The minimum, written in terms of the (constant) average of $\hat{Z}$, is

$$\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle = \left(\hbar^2\omega_0^2 e^{-2\gamma t} + \frac{\hbar^2\gamma^2/4}{e^{2\gamma t}-1}\right)\left(1-\langle\hat{Z}\rangle^2\right), \tag{13}$$

so that (4) becomes, in terms of elliptic integrals of the second kind, $E(y,k)$,

$$\begin{aligned} \arccos\sqrt{F_B} &\leq \sqrt{1-\langle\hat{Z}\rangle^2}\sqrt{r^2+\tfrac{1}{4}} \\ &\times \left[E\left(\frac{\pi}{2}, \frac{r}{\sqrt{r^2+\frac{1}{4}}}\right) - E\left(\arcsin e^{-\gamma\tau}, \frac{r}{\sqrt{r^2+\frac{1}{4}}}\right)\right], \end{aligned} \tag{14}$$

with $r := \omega_0/\gamma$. Eq. (14) consistently guarantees the eigenstates of $\hat{Z}$ not to evolve. The corresponding bound for the fidelity F_B is compared to an exact calculation in Fig. 1, which shows that the bound stays close to the exact result up to the first minimum of the latter.

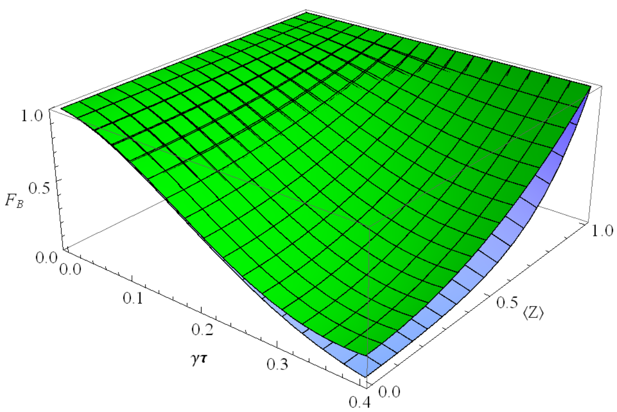

Figure 1: Comparison of the evolution of the bound on fidelity F_B (lower surface) as a function of the dimensionless time $\gamma\tau$ to an exact calculation (upper surface) for different values of $\langle\hat{Z}\rangle$ (different initial states), with r =4.

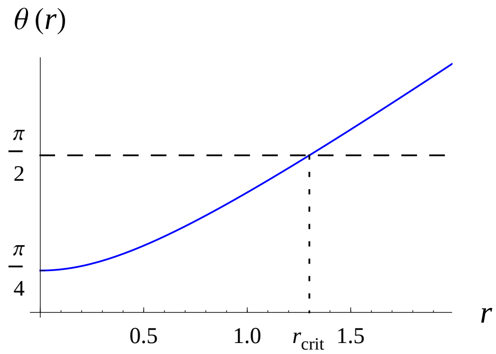

Figure 2: Bound $\theta(r)$ for $\arccos\sqrt{F_B}$ corresponding to single-qubit Markovian dephasing, given by the right-hand side of (14) for evolution time $\tau \to \infty$ and $\langle\hat{Z}\rangle = 0$. Values under $\pi/2$ indicate regions where the bound guarantees that no state turns orthogonal to itself.

In the extreme cases $\omega_0 = 0$ and $\gamma = 0$, (13) yields simple analytical expressions for the bound on the minimal time, which correspond to the exact results when $\langle\hat{Z}\rangle = 0$. For the former,

$$\tau \geq \frac{1}{\gamma}\ln\sec\left(\frac{2\arccos\sqrt{F_B}}{\sqrt{1-\langle\hat{Z}\rangle^2}}\right) \tag{15}$$

and for the latter,

$$\tau \geq \frac{1}{\omega_0}\frac{\arccos\sqrt{F_B}}{\sqrt{1-\langle\hat{Z}\rangle^2}}. \tag{16}$$

The bound on the minimal time behaves quite differently in these two situations: for $\omega_0 = 0$ the limit guarantees that the evolved and initial states never become orthogonal, since even for $\tau \to \infty$, $\arccos\sqrt{F_B}$ reaches, at most, $\pi/4$; whereas for $\gamma = 0$ the bound does not lead to a similar restriction for any state other than the eigenstates of $\hat{Z}$. There is a transition between these two regimes, as seen in Fig. 2, which displays the r.h.s. in (14) with $\tau \to \infty$, $\langle\hat{Z}\rangle = 0$, as a function of r. There clearly is a region $r < r_{\text{crit}}$ where the $\arccos\sqrt{F_B}$ does not reach $\pi/2$ for any finite τ, hence the evolved and initial states never become orthogonal (for any initial state). This exclusion window – calculations give $r_{\text{crit}} \simeq 1.3$ – yields a simple criterion for defining the regimes of strong ($r > r_{\text{crit}}$) and weak ($r < r_{\text{crit}}$) dephasing; exclusion windows for reaching given finite fidelities can be calculated in an analogous way.

Minimum evolution time and entanglement

We now investigate the effect of subsystem correlations on the evolution speed of a compound system. We consider the Markovian dephasing of an N-qubit system where each qubit interacts only with its own environment, as described by (10), and compare how different initial-state correlations (possibly entanglement) affect the evolution speed. The evolution operator is

$$\hat{u}_E(t)\prod_{i=1}^{N}\hat{U}_{S,E}^{(i)} = \hat{u}_E(t)\prod_{i=1}^{N}\Big\{e^{-i\omega_0 t\hat{Z}_i} \times e^{-i\arccos\sqrt{P(\gamma t)}\hat{Z}_i\hat{Y}_i^{(E)}}\Big\}. \tag{17}$$

Since $\hat{h}_E(t)$ defined by (6) now belongs to a $2^N \times 2^N$ space, its most general form is rather cumbersome for large values of N. We resort instead to minimization

over a three-parameter family, hinging on the symmetry of the system:

$$\hat{h}_E(t) = \alpha(t)\sum_{i=1}^{N}\hat{X}_i^{(E)} + \beta(t)\sum_{i=1}^{N}\hat{Y}_i^{(E)} + \delta(t)\sum_{i=1}^{N}\hat{Z}_i^{(E)}, \tag{18}$$

where $\alpha(t)$, $\beta(t)$, and $\delta(t)$ are optimization variables. We get then

$$\arccos\sqrt{F_B} \leq \sqrt{1-\langle\hat{\mathcal{Z}}\rangle^2} \tag{19}$$
$$\times \int_0^{\gamma\tau}\sqrt{r^2\frac{N^2 q}{Nq(e^{2u}-1)+1} + \frac{1}{4}\frac{N}{e^{2u}-1}}du,$$

where $q := \langle\Delta\hat{\mathcal{Z}}^2\rangle/(1-\langle\hat{\mathcal{Z}}\rangle^2)$, $\hat{\mathcal{Z}} = \sum_j \hat{Z}_j/N$, and the averages are taken in the initial state. We note that $0 \leq q \leq 1$; for a separable state, $q \leq 1/N$ (equality if symmetrical on the N qubits). The values $q = 1$ and $\langle\hat{\mathcal{Z}}\rangle = 0$, achievable for GHZ states $[|0\ldots0\rangle + e^{i\phi}|1\ldots1\rangle]/\sqrt{2}$, yield a lower bound valid for any possible state. For $q = 1$, the bound on the minimum evolution time scales as $\tau \sim 1/N$ throughout, see Supplemental Material.

For separable states, on the other hand, the lower bound goes from a $\tau \sim 1/\sqrt{N}$ dependence for $\gamma\sqrt{N} \ll \omega_0$ to $\sim 1/N$ for $\gamma\sqrt{N} \gg \omega_0$, as shown in Fig. 3 and in the Supplemental Material.

This is a striking result, clearly distinct from the one corresponding to unitary evolution. It has already been seen in the literature [23–27] that for unitary evolution entanglement is a resource that enhances computation time from a $\tau \sim 1/\sqrt{N}$ scaling (separable, slow state) to $\tau \sim 1/N$ (entangled, fast state). However, for the nonunitary evolution here considered, the minimum evolution time for separable states, while scaling with $1/\sqrt{N}$ for small N, eventually scales as $1/N$ for $\gamma\sqrt{N} \gg \omega_0$. This result, which exemplifies the usefulness of our approach, shows that, in the presence of decoherence, the $1/N$ behavior may be attained also for separable states, and no matter how small is the decoherence rate γ. One should note that the lower bounds obtained here bypass solving the actual equations of motion of the system, and are valid for any initial state.

This novel result can be corroborated for the two particular initial states above via direct calculations.

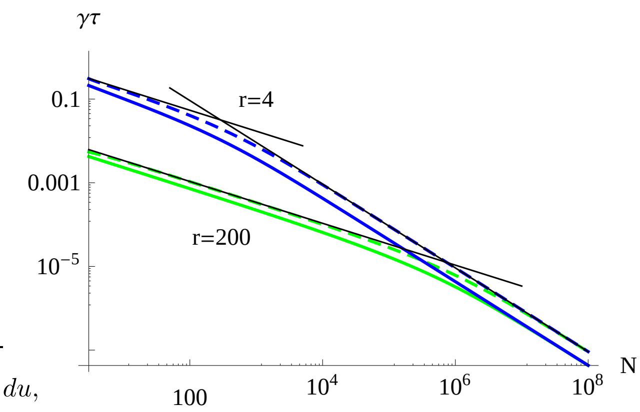

Figure 3: (Color online) Lower bound (solid curves) on time for separable, symmetric state with $\langle\hat{\mathcal{Z}}\rangle = 0$ to reach $F_B = 1\%$, measured in dimensionless units $\gamma\tau$ as a function of number of qubits N, calculated numerically from (19). Results from exact calculations (21) are plotted for comparison (dashed curves). For the black (blue) curves, $r = 4$, for the light gray (green) curves, $r = 200$; the asymptotes are proportional to $1/N$ and $1/\sqrt{N}$, see Supplemental Material.

For the GHZ state, the exact fidelity at time τ is

$$F_B = \frac{1 + e^{-N\gamma\tau}\cos 2N\omega_0\tau}{2} \tag{20}$$

and clearly $\tau \sim 1/N$ for any regime. Meanwhile, for the separable, symmetric state with each qubit on the equator of its Bloch sphere ($\langle\hat{Z}\rangle = 0$),

$$F_B = \frac{1}{2^N}\left(1 + e^{-\gamma\tau}\cos 2\omega_0\tau\right)^N, \tag{21}$$

and a suitable expansion (Supplemental Material) readily shows that for $\gamma\sqrt{N} \ll \omega_0$, $\tau \sim 1/\sqrt{N}$, while for $\gamma\sqrt{N} \gg \omega_0$, $\tau \sim 1/N$. The same transition to fast behavior is then observed, corroborating our previous result, see Fig. 3. It should be noted that there is actually another constraint to the fast regime, given by the breakdown of the Markovian approximation for extremely short times.

Conclusion

We have developed a geometrical approach that leads to an attainable lower bound for the minimal evolution time of dynamical systems. This bound applies to both unitary and nonunitary processes, and is obtained by comparing the actual path followed by the system in state space with the distance between the initial and final states along a geodesic path, defined by a metric that is expressed in terms of the Bures fidelity. Whenever the evolution between two states is along this geodesic, the bound is tight. Furthermore, it encompasses several special cases discussed in the literature, including unitary evolutions and mixed initial states.

The usefulness of this bound is exemplified by considering typical nonunitary quantum channels. For the amplitude channel, it leads to a tight bound, which evidentiates that the evolution between the initial and final orthogonal pure states is along a geodesic path though mixed states. For a dephasing channel, it yields very good lower bounds for the mininal evolution time between two non-orthogonal states. For N-qubit dephasing, the evolution speed-up due to entanglement of its subsystems, previously demonstrated for unitary evolution, is shown to hold, in the nonunitary case, also for separable states.

Our general result allows the estimation of the impact of the environment on the speed of quantum computation and information processing. It is also relevant for the estimation of thermalization and decoherence times.

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[45] These results remain valid if a time-dependent relative phase is added between the two states on the right-hand side of (7b).

Supplemental material

A. Geometric derivation of the general bound

Let $D(F_B)$ be any function dependent only on the Bures fidelity F_B defined in (1). We denote by $F_B(t_1, t_2)$ the fidelity between the states of the same system at times t_1, t_2, i.e. a shorthand notation for $F_B\left[\hat{\rho}(t_1), \hat{\rho}(t_2)\right]$; $D(t_1, t_2)$ will analogously be a shorthand notation for $D\{F_B\left[\hat{\rho}(t_1), \hat{\rho}(t_2)\right]\}$. Here we consider dynamical evolutions of the system and functions $D(F_B)$ so that $D(t_1, t_2)$ can be considered as a piecewise smooth metric on the space of quantum systems, which implies that $\partial D(t,x)/\partial x$ is different from zero when $x \to t$. Dividing a given time interval into an arbitrary number of smaller ones, and using the triangle inequality for distances, one gets

$$D(0,\tau) \leq \sum_{i=1}^{n} D[(i-1)\Delta t, i\Delta t], \tag{S1}$$

where $\Delta t = \tau/n$. An expansion in the second argument of the i-th term of the sum around $(i-1)\Delta t$ yields

$$D[(i-1)\Delta t, i\Delta t] = \left.\frac{\partial D[(i-1)\Delta t, x]}{\partial x}\right|_{x\to(i-1)\Delta t} \Delta t + \mathcal{O}(\Delta t^2), \tag{S2}$$

so that, when $n \to \infty$, one finds

$$D(0,\tau) \leq \int_0^\tau \left.\frac{\partial D[t,x]}{\partial x}\right|_{x\to t} dt. \tag{S3}$$

The integrand can be calculated using the chain rule,

$$\left.\frac{\partial D(t,x)}{\partial x}\right|_{x\to t} = \left[\frac{dD(F_B)}{dF_B}\frac{\partial F_B(t,x)}{\partial x}\right]_{x\to t}. \tag{S4}$$

Using Eq. (2) of the main text, we see that $\partial F_B(t,x)/\partial x$ tends to zero when $x \to t$. Since $D(t,x)$ is a piecewise smooth metric as a function of x, $\partial D(t,x)/\partial x|_{x\to t}$ is finite, and hence $dD(F_B)/dF_B$ must diverge in the $x \to t$ limit. This indeterminacy can be removed with l'Hôpital's rule,

$$\left.\frac{\partial D(t,x)}{\partial x}\right|_{x\to t} = \left.\frac{\dfrac{\partial F_B(t,x)}{\partial x}}{\dfrac{1}{dD(F_B)/dF_B}}\right|_{x\to t} = \left.\frac{\dfrac{\partial^2 F_B(t,x)}{\partial x^2}}{\dfrac{d}{dF_B}\left[\dfrac{1}{dD(F_B)/dF_B}\right]\dfrac{\partial F_B(t,x)}{\partial x}}\right|_{x\to t}. \tag{S5}$$

While the numerator is proportional to $\mathcal{F}_Q(t)$ due to (2), the denominator can be written, by multiplying and dividing by dD/dF_B, as

$$\left.\left(-\frac{1}{\left[dD(F_B)/dF_B\right]^3}\frac{d^2D(F_B)}{dF_B^2}\right)\left(\frac{dD(F_B)}{dF_B}\frac{\partial F_B(t,x)}{\partial x}\right)\right|_{x\to t}. \tag{S6}$$

The first factor in parentheses can be calculated independently of t, x by replacing the limit $x \to t$ by $F_B \to 1$ since the metric D only depends on t, x through $F_B(t,x)$. This first factor is actually proportional

to the curvature of the curve $D(F_B)$ in $F_B = 1$. The second factor in parentheses is simply a recurrence of $\partial D(t,x)/\partial x|_{x\to t}$, the term we are calculating. Substituting in (S5) and rearranging the terms, one finds

$$\left[\frac{\partial D(t,x)}{\partial x}\right]^2_{x\to t} = \left.\frac{2\left[\dfrac{dD(F_B)}{dF_B}\right]^3}{\dfrac{d^2D(F_B)}{dF_B^2}}\right|_{F_B\to 1} \frac{\mathcal{F}_Q(t)}{4}. \tag{S7}$$

Taking the square root of the r.h.s. of the above equation into (S3), and rearranging the terms, one obtains the general bound (3), valid for any piecewise smooth metric $D[F_B(t,x)]$ depending only on the Bures fidelity $F_B(t,x)$.

We show now, via purification techniques, that the metric $D(F_B) = 2\arccos\sqrt{F_B}$ leads to the largest lower bound on τ. This is done by demonstrating that there exists a path between any two given states $\hat{\rho}_0$ and $\hat{\rho}_\tau$, along a geodesic, so that

$$\int_{\hat{\rho}_0}^{\hat{\rho}_\tau} \sqrt{\mathcal{F}_Q(t)}dt = 2\arccos\sqrt{F_B(\hat{\rho}_0,\hat{\rho}_\tau)}\,. \tag{S8}$$

We start from the particularization of the above relation for pure states under unitary evolution, which was demonstrated in [13]:

$$\int_{|\psi_0\rangle}^{|\psi_\tau\rangle} \sqrt{4\langle\Delta\hat{H}^2(t)\rangle}dt = 2\arccos\sqrt{F_B(|\psi_0\rangle\langle\psi_0|,|\psi_\tau\rangle\langle\psi_\tau|)}, \tag{S9}$$

where $|\psi_\tau\rangle$ are pure states in a projective Hilbert space and $\hat{H}(t)$ is the Hamiltonian generating the evolution from $|\psi_0\rangle$ to $|\psi_\tau\rangle$ along a geodesic.

Notice now that, to any given initial and final states $\hat{\rho}_0$ and $\hat{\rho}_\tau$ of a system S, one can associate respective purifications $|\Psi_0\rangle$ and $|\Psi_\tau\rangle$, so that

$$|\langle\Psi_0|\Psi_\tau\rangle|^2 = F_B(\hat{\rho}_0,\hat{\rho}_\tau). \tag{S10}$$

For these specific purifications, one can now design a Hamiltonian $\hat{H}(t)$ that connects $|\Psi_0\rangle$ to $|\Psi_\tau\rangle$ along a geodesic in the enlarged state space corresponding to $S+E$. This evolution corresponds to an evolution, in general non-unitary, of S, described by the state $\hat{\rho}(t)$, which connects $\hat{\rho}_0$ to $\hat{\rho}_\tau$. The corresponding quantum Fisher information satisfies, according to [43], the inequality $4\langle\Delta\hat{H}^2(t)\rangle \geq \mathcal{F}_Q(t)$, where the variance is calculated in the purification $|\Psi(0)\rangle$. Taking this inequality and (S10) into (S9) leads to

$$\int_{\hat{\rho}_0}^{\hat{\rho}_\tau} \sqrt{\mathcal{F}_Q(t)}dt \leq 2\arccos\sqrt{F_B(\hat{\rho}_0,\hat{\rho}_\tau)}\,. \tag{S11}$$

Since, from (4), the l.h.s. of the above equation must be larger or equal to the r.h.s, it follows that the equality sign prevails for the chosen evolution. This implies that the evolution of $\hat{\rho}(t)$ is also along a geodesic in the corresponding state space.

B. Amplitude-damping channel: saturation of the bound for general $P(t)$

For the evolution operator $\hat{U}_{S,E}(t)$ of the amplitude-damping channel displayed in (8), one has

$$\hat{H}_{S,E}(t) = \hbar \left(\hat{\sigma}_+\hat{\sigma}_-^{(E)} + \hat{\sigma}_-\hat{\sigma}_+^{(E)}\right) \frac{d\Theta(t)}{dt}, \tag{S11}$$

$$\langle \Delta \hat{H}^2_{S,E}(t)\rangle = \hbar^2 \langle \hat{\sigma}_+\hat{\sigma}_-\rangle \left(\frac{d\Theta(t)}{dt}\right)^2. \tag{S11}$$

For S initially in the ground state, the trivially saturated bound $F_B \geq 1$ is obtained, whereas an initially excited state of S yields

$$\arccos\sqrt{F_B} \leq \int_0^\tau \left|\frac{d\Theta(t)}{dt}\right| dt = \int_0^\tau \left|\frac{d\arccos\sqrt{P(t)}}{dt}\right| dt, \tag{S12}$$

where we have used $\Theta(t) = \arccos\sqrt{P(t)}$. Hence, for any monotonically decreasing $P(t)$,

$$\arccos\sqrt{F_B} \leq \arccos\sqrt{P(\tau)}, \tag{S13}$$

a bound which is clearly saturated by the actual fidelity $F_B = P(t)$.

C. Dephasing channel: derivation of bound

It is straightforward to show from the definitions of $\hat{H}_{S,E}(t)$ and $\hat{\mathcal{H}}_{S,E}(t)$ that

$$\hat{\mathcal{H}}_{S,E}(t) = \hat{H}_{S,E}(t) + \hat{h}'(t), \tag{S14}$$

where $\hat{h}'(t) := \hat{U}^\dagger_{S,E}(t)\hat{h}_E(t)\hat{U}_{S,E}(t)$. The variance to be minimized is $\langle \Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle$, which can be cast in the form

$$\langle \Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle = \langle \Delta\hat{H}^2_{S,E}(t)\rangle + \langle \Delta\hat{h}^2_E(t)\rangle + 2\Re[\langle \hat{h}'(t)\hat{H}_{S,E}(t)\rangle - \langle \hat{h}'(t)\rangle\langle \hat{H}_{S,E}(t)\rangle]. \tag{S15}$$

Let $|\psi_0\rangle|0\rangle_E$ be the initial state of $S+E$. The N-qubit dephasing is described by (17), from which one gets

$$\hat{H}_{S,E}(t) = \hbar\omega_0 N\hat{\mathcal{Z}} + \frac{\hbar\gamma/2}{\sqrt{e^{2\gamma t}-1}} \sum_{i=1}^N \hat{Z}_i\hat{Y}_i^{(E)}, \tag{S15}$$

$$\langle \hat{H}_{S,E}(t)\rangle = \hbar\omega_0\langle\hat{\mathcal{Z}}\rangle, \tag{S15}$$

which imply that

$$\langle \Delta\hat{H}^2_{S,E}(t)\rangle = N^2\hbar^2\omega_0^2\langle\Delta\hat{\mathcal{Z}}^2\rangle + \frac{N\hbar^2\gamma^2/4}{e^{2\gamma t}-1}. \tag{S16}$$

From (18),

$$\langle \hat{h}'(t)\rangle = \delta(t)N\left(2P(t)-1\right) - \alpha(t)N\langle\hat{\mathcal{Z}}\rangle 2\sqrt{P(t)}\sqrt{1-P(t)}, \tag{S17}$$

where $P(t) := (1+e^{-\gamma t})/2$, and

$$\begin{aligned}\langle \Delta\hat{h}'^2(t)\rangle =& N(\alpha^2(t)+\beta^2(t)+\delta^2(t)) + \alpha^2(t)\left(N^2\langle\Delta\hat{\mathcal{Z}}^2\rangle - N\right)4P(t)\left(1-P(t)\right)\\ &- N\delta^2(t)\left(2P(t)-1\right)^2 + 2\alpha(t)\delta(t)N\langle\hat{\mathcal{Z}}\rangle 2\sqrt{P(t)}\sqrt{1-P(t)}\left(2P(t)-1\right).\end{aligned} \tag{S18}$$

From the previous equations,

$$2\Re[\langle \hat{h}'(t)\hat{H}_{S,E}(t)\rangle - \langle \hat{h}'(t)\rangle\langle \hat{H}_{S,E}(t)\rangle] = -2\alpha(t)\hbar\omega_0 N^2\langle\Delta\hat{\mathcal{Z}}^2\rangle 2\sqrt{P(t)}\sqrt{1-P(t)} + \frac{N\hbar\gamma\beta(t)}{\sqrt{e^{2\gamma t}-1}}\langle\hat{\mathcal{Z}}\rangle, \tag{S19}$$

and $\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle$, according to (S15), is the sum of (S16), (S18), and (S19).

One now minimizes over $\alpha(t)$, $\beta(t)$, $\delta(t)$ for each t. From $\partial\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle/\partial\beta = 0$, one gets

$$\beta(t) = -\frac{\hbar\gamma}{2\sqrt{e^{2\gamma t}-1}}\langle\hat{\mathcal{Z}}\rangle. \tag{S20}$$

Conditions $\partial\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle/\partial\alpha = 0$ and $\partial\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle/\partial\delta = 0$ lead to a linear system of equations, with solutions

$$\alpha(t) = \frac{\hbar\omega_0 e^{\gamma t}\sqrt{e^{2\gamma t}-1}Nq}{1+(e^{2\gamma t}-1)Nq}, \tag{S20}$$

$$\delta(t) = -\frac{\hbar\omega_0 e^{2\gamma t}\langle\hat{\mathcal{Z}}\rangle Nq}{1+(e^{2\gamma t}-1)Nq}, \tag{S20}$$

where $q := \langle\Delta\hat{\mathcal{Z}}^2\rangle/(1-\langle\hat{\mathcal{Z}}\rangle^2)$. By replacing (S20), (S20), and (S20) into (S16), (S18), and (S19), one finds

$$\langle\Delta\hat{\mathcal{H}}^2_{S,E}(t)\rangle = \sqrt{1-\langle\hat{\mathcal{Z}}\rangle^2}\left\{\hbar^2\omega_0^2\frac{N^2q}{Nq(e^{2\gamma t}-1)+1} + \frac{N\hbar^2\gamma^2/4}{e^{2u}-1}\right\}, \tag{S21}$$

and (19) follows due to (4). The single-qubit result (13) is recovered by taking $N = 1$ (note that $q = 1$ in this case).

D. Dephasing channel: N-dependence of the most general bound

The most general bound for N qubits under Markovian dephasing, attained with parameters corresponding to the GHZ state, is

$$\arccos\sqrt{F_B} \leq \sqrt{N}\int_0^{\gamma\tau}\sqrt{r^2\frac{N}{N(e^{2u}-1)+1} + \frac{1/4}{e^{2u}-1}}du. \tag{S22}$$

An equally general, albeit slightly larger, upper bound for $\arccos\sqrt{F_B}$ leads to an analytical expression for the bound on τ. It is found by considering $N(e^{2\gamma\tau}-1) \gg 1$. One gets then

$$\sqrt{N}\sqrt{r^2+1/4}\arctan\sqrt{e^{2\gamma\tau}-1} \geq \arccos\sqrt{F_B}, \tag{S23}$$

which yields, for $N \gg 1$,

$$\tau \geq \frac{1}{N}\frac{\arccos^2\sqrt{F_B}}{\gamma(r^2+1/4)}. \tag{S24}$$

This expression explicitly exhibits a $\tau \sim 1/N$ dependence. An alternative estimate, which leads to a better approximation of the integral in (S22), is found in the $r \gg 1$ limit, yielding, to lowest order in $1/N$:

$$\tau \geq \frac{1}{N}\frac{\arccos\sqrt{F_B}}{\omega_0}\left(1+\frac{\arccos\sqrt{F_B}}{2r}\right). \tag{S25}$$

Fig.S1 displays the numerically calculated bound, which is compared to the approximation (S25) .

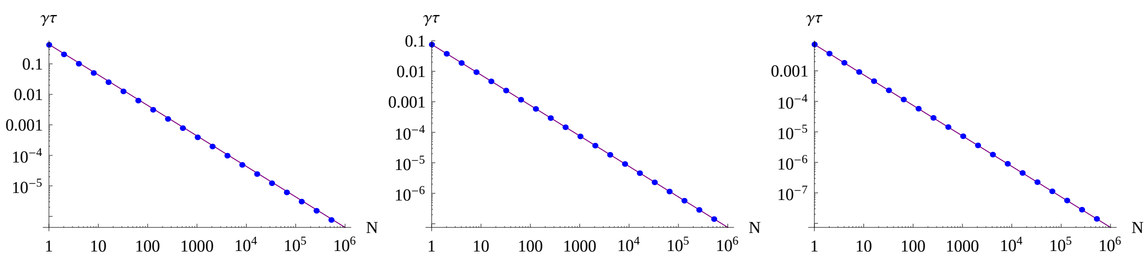

Figure S1: Most general lower bound on time $\gamma\tau$ for state to reach $F_B = 1\%$, calculated numerically from (S22), as a function of N, with $r = 4$, 20 and 200, respectively (error bars smaller than symbols). The straight line, proportional to $1/N$, obeys (S25).

E. Dephasing channel: N-dependence of the bound for separable, symmetric states

From (19), the bound for separable states can be written in terms of elliptic functions of the second kind $E(y,k)$,

$$\arccos\sqrt{F_B} \leq \sqrt{1-\langle\hat{Z}\rangle^2}\sqrt{N}\sqrt{r^2+\tfrac{1}{4}}\left[E\left(\frac{\pi}{2},\frac{r}{\sqrt{r^2+\frac{1}{4}}}\right)-E\left(\arcsin e^{-\gamma\tau},\frac{r}{\sqrt{r^2+\frac{1}{4}}}\right)\right]. \tag{S26}$$

An expansion to lowest order on $\gamma\tau$ yields $\tau \sim 1/N$:

$$\tau \geq \frac{1}{N}\frac{2\arccos^2\sqrt{F_B}}{\gamma(1-\langle\hat{Z}\rangle^2)}. \tag{S27}$$

For $r \gg 1$, one gets instead, from (S26),

$$\arccos\sqrt{F_B} \leq \sqrt{1-\langle\hat{Z}\rangle^2}\sqrt{N}r(1-e^{-\gamma\tau}), \tag{S28}$$

which leads to a $\tau \sim 1/\sqrt{N}$ dependence,

$$\tau \geq \frac{1}{\sqrt{N}}\frac{\arccos\sqrt{F_B}}{\omega_0\sqrt{1-\langle\hat{Z}\rangle^2}}. \tag{S29}$$

An estimate of the transition between (S27) and (S29) is obtained by equating the two respective contributions to the bound on $\arccos\sqrt{F_B}$. This leads to

$$\gamma\tau_{\mathrm{tr}} \simeq 1/(2r^2) \tag{S29}$$

$$\sqrt{N_{\mathrm{tr}}} \simeq r\frac{2\arccos\sqrt{F_B}}{\sqrt{1-\langle\hat{Z}\rangle^2}}, \tag{S29}$$

resulting in a $\sqrt{N_{\mathrm{tr}}} \sim r$ scaling.

F. Dephasing channel: N-dependence for separable, symmetric state via exact calculations

The fidelity of the separable, symmetric state of N qubits, each initially on the equator of its Bloch sphere, given by (21), can be rewritten and expanded as

$$e^{-\gamma\tau}\cos(2\omega_0\tau) = 2{F_B}^{1/N} - 1, \tag{S29}$$

$$\left(1 - \gamma\tau + \frac{\gamma^2\tau^2}{2} + \mathcal{O}(\gamma^3\tau^3)\right)\left(1 - 2\omega_0^2\tau^2 + \mathcal{O}(\omega_0^4\tau^4)\right) = 1 - 2\frac{\ln(1/F_B)}{N} + \mathcal{O}(1/N^2), \tag{S29}$$

where we make use of the fact that τ vanishes for increasing values of N. The above expression can be simplified to

$$\gamma\tau + \frac{4\omega_0^2 - \gamma^2}{2}\tau^2 + \mathcal{O}(\gamma^3\tau^3, \omega^3\tau^3) = 2\frac{\ln(1/F_B)}{N} + \mathcal{O}(1/N^2). \tag{S30}$$

The two regimes can be seen in the above equation: for large values of N the first term of the left-hand side is dominant, yielding a $\tau \sim 1/N$ dependence; for smaller values of N, the second term is dominant, and $\tau \sim 1/\sqrt{N}$. We can estimate when this transition occurs by finding the value of τ that leads to equal contributions from both terms of the left-hand side of (S30), which is $\tau_{\mathrm{tr}} = 2\gamma/(4\omega_0^2 - \gamma^2)$. The corresponding value of N is

$$N_{\mathrm{tr}} = 2\ln(1/F_B)(r^2 - 1/4), \tag{S31}$$

so that for $r \gg 1$ the transition happens around a value N proportional to r^2.